

Vector Analysis*

Elementary GFD Survival Kit

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General relations

$$\text{Relative derivative } \frac{d(f^\alpha g^\beta \dots h^\gamma)}{f^\alpha g^\beta \dots h^\gamma} = \alpha \frac{df}{f} + \beta \frac{dg}{g} + \dots + \gamma \frac{dh}{h}$$

Vector Algebra

$\mathbf{a}, \mathbf{b}, \mathbf{c} \dots$ Vectors, $\alpha \dots$ Scalar

$$\begin{aligned}\alpha(\mathbf{a} + \mathbf{b}) &= \alpha\mathbf{a} + \alpha\mathbf{b} \\ \mathbf{a} \cdot \mathbf{b} &= \mathbf{b} \cdot \mathbf{a} \\ \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \\ \alpha(\mathbf{a} \cdot \mathbf{b}) &= (\alpha\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\alpha\mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\alpha \\ \mathbf{a} \times \mathbf{b} &= -\mathbf{b} \times \mathbf{a} \\ \mathbf{a} \times \mathbf{a} &= \mathbf{0} \\ \alpha(\mathbf{a} \times \mathbf{b}) &= (\alpha\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\alpha\mathbf{b}) = (\mathbf{a} \times \mathbf{b})\alpha \\ \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \\ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \\ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \\ (\text{mnemonic : abc}) &= bac - cab \\ (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \\ (\mathbf{a} \times \mathbf{b})^2 &= \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a} \cdot \mathbf{b})^2\end{aligned}$$

Vector Analysis

$\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{P}, \mathbf{v} \dots$ (3-dim.) vector fields, $\Phi, \Psi, \chi, \psi \dots$ scalar fields

Differential Operations

$$\begin{aligned}\nabla(\Phi + \Psi) &= \nabla\Phi + \nabla\Psi \\ \nabla(\Phi\Psi) &= \Psi\nabla\Phi + \Phi\nabla\Psi \\ \nabla \cdot (\mathbf{A} + \mathbf{B}) &= \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \\ \nabla \cdot (\mathbf{AB}) &= (\nabla\mathbf{A}) \cdot \mathbf{B} + (\nabla\mathbf{B}) \cdot \mathbf{A} \\ \nabla \cdot (\Phi\mathbf{A}) &= (\nabla\Phi) \cdot \mathbf{A} + \Phi(\nabla \cdot \mathbf{A}) \\ \nabla \times (\mathbf{A} + \mathbf{B}) &= \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \\ \nabla \times (\Phi\mathbf{A}) &= \nabla\Phi \times \mathbf{A} + \Phi(\nabla \times \mathbf{A})\end{aligned}$$

$$\begin{aligned}\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \\ \nabla \times (\mathbf{A} \times \mathbf{B}) &= (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - \\ &\quad - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) \\ \nabla(\mathbf{A} \cdot \mathbf{B}) &= (\nabla\mathbf{A}) \cdot \mathbf{B} + (\mathbf{A}\nabla) \cdot \mathbf{B} = \\ &\quad + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B}) \\ \nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \\ \text{Weber transformation ... } \mathbf{v} \cdot \nabla \mathbf{v} &= \frac{1}{2} \nabla \mathbf{v}^2 + (\nabla \times \mathbf{v}) \times \mathbf{v} \\ &= \frac{1}{2} \nabla \mathbf{v}^2 + \vec{\zeta} \times \mathbf{v} \\ \nabla(\mathbf{A} \cdot \mathbf{B}) &= (\nabla\mathbf{A}) \cdot \mathbf{B} + (\mathbf{A} \cdot \nabla)\mathbf{B} \\ \nabla \times (\nabla\Phi) &= \mathbf{0} \\ \nabla \cdot (\nabla \times \mathbf{A}) &= 0 \\ \mathbf{A} \times (\nabla \times \mathbf{B}) - (\mathbf{A} \times \nabla) \times \mathbf{B} &= \mathbf{A} \nabla \cdot \mathbf{B} - (\mathbf{A} \nabla) \cdot \mathbf{B} \\ \text{position vector (cart.) ... } \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ \nabla \cdot \mathbf{r} &= 3 \\ \nabla \times \mathbf{r} &= \mathbf{0}\end{aligned}$$

fundamental theorem of vector analysis	rotation free (with divergence)	divergence free (with rotation)
3-dim. \mathbf{A}	$\underbrace{\nabla \chi}_{\text{scalar potential}}$	$\underbrace{\nabla \times \mathbf{P}}_{\text{vector potential}}$
2-dim. \mathbf{A}_h	$\nabla_h \chi$	$\mathbf{k} \times \nabla_h \psi$

Integral Theorems

$$\begin{aligned}\text{Gauss } \iiint_V \nabla \cdot \mathbf{A} dV &= \iint_S \mathbf{A} \cdot \mathbf{n} d\sigma \\ \text{2D-Gauss } \iint_S \nabla_h \cdot \mathbf{A} d\sigma &= \oint_C \mathbf{A} \cdot d\mathbf{r} \\ \text{Stokes } \iint_S \nabla \times \mathbf{A} \cdot \mathbf{n} d\sigma &= \oint_C \mathbf{A} \cdot d\mathbf{r} \\ \text{2D-Stokes } \iint_S \mathbf{k} \cdot (\nabla_h \times \mathbf{A}_h) d\sigma &= \oint_C \mathbf{A}_h \cdot d\mathbf{r} \\ \text{or: Green } \iint \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dx dy &= \oint_C (A_x dx + A_y dy)\end{aligned}$$

Coordinate Systems

Cartesian Coordinates

$$\begin{aligned}\text{unity vectors } \mathbf{i}, \mathbf{j}, \mathbf{k}: \quad \mathbf{i} \cdot \mathbf{i} = 1 \text{ etc...}, \quad \mathbf{i} \times \mathbf{i} = \mathbf{0} \text{ etc...}, \\ \mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \text{arbitr. vector } \mathbf{A} &= A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}\end{aligned}$$

$$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ \nabla \times \mathbf{A} &= \mathbf{i} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{j} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\ \nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\end{aligned}$$

Cylindrical Coordinates

radius $r > 0$, azimuth $0 \leq \lambda < 2\pi$, height z :

$$x = r \cos \lambda, \quad y = r \sin \lambda, \quad z = z$$

$$\begin{aligned}\nabla f &= \mathbf{e}_r \frac{\partial f}{\partial r} + \frac{\mathbf{e}_\lambda}{r} \frac{\partial f}{\partial \lambda} + \mathbf{e}_z \frac{\partial f}{\partial z} \\ \nabla \cdot \mathbf{A} &= \frac{1}{r} \frac{\partial(rA_\lambda)}{\partial r} + \frac{1}{r} \frac{\partial A_\lambda}{\partial \lambda} + \frac{\partial A_z}{\partial z} \\ \nabla \times \mathbf{A} &= \mathbf{e}_r \left(\frac{1}{r} \frac{\partial A_z}{\partial r} - \frac{\partial A_\lambda}{\partial z} \right) + \\ &\quad + \mathbf{e}_\lambda \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \\ &\quad + \mathbf{e}_z \left(\frac{1}{r} \frac{\partial(rA_\lambda)}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \lambda} \right) \\ \nabla^2 f &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \lambda^2} + \frac{\partial^2 f}{\partial z^2}\end{aligned}$$

Spherical coordinates

radius $r > 0$, longitude $0 \leq \lambda < 2\pi$, latitude φ :

$$x = r \cos \varphi \cos \lambda, \quad y = r \cos \varphi \sin \lambda, \quad z = r \sin \varphi$$

$$\begin{aligned}\nabla f &= \frac{\mathbf{e}_\lambda}{r \cos \varphi} \frac{\partial f}{\partial \lambda} + \frac{\mathbf{e}_\varphi}{r} \frac{\partial f}{\partial \varphi} + \mathbf{e}_r \frac{\partial f}{\partial r} \\ \nabla \cdot \mathbf{A} &= \frac{1}{r \cos \varphi} \frac{\partial A_\lambda}{\partial \lambda} + \frac{1}{r \cos \varphi} \frac{\partial}{\partial \varphi} (\cos \varphi A_\varphi) + \frac{1}{r^2} \frac{\partial (A_r r^2)}{\partial r} \\ \nabla \times \mathbf{A} &= \frac{1}{r^2 \cos \varphi} \begin{vmatrix} \mathbf{e}_\lambda r \cos \varphi & \mathbf{e}_\varphi r & \mathbf{e}_r \\ \partial/\partial \lambda & \partial/\partial \varphi & \partial/\partial r \\ A_\lambda r \cos \varphi & A_\varphi r & A_r \end{vmatrix} = \\ &= \frac{1}{r^2 \cos \varphi} \left\{ \frac{\partial A_r}{\partial \varphi} - \frac{\partial(rA_\varphi)}{\partial r} \right\} r \cos \varphi \mathbf{e}_\lambda + \\ &+ \frac{1}{r^2 \cos \varphi} \left\{ \frac{\partial(r \cos \varphi A_\lambda)}{\partial r} - \frac{\partial A_r}{\partial \lambda} \right\} r \mathbf{e}_\varphi + \\ &+ \frac{1}{r^2 \cos \varphi} \left\{ \frac{\partial(rA_\varphi)}{\partial \lambda} - \frac{\partial(r \cos \varphi A_\lambda)}{\partial \varphi} \right\} \mathbf{e}_r\end{aligned}$$

$$\nabla^2 f = \frac{1}{r^2 \cos \varphi} \frac{\partial^2 f}{\partial \lambda^2} + \frac{1}{r^2 \cos \varphi} \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial f}{\partial \varphi} \right) + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right)$$